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Mathematical model for thermoelasticity of pre-stressed solids and problems for non-destructive determination of stress-tensor fields

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Abstract A mathematical model for thermoelastic processes in a piecewise–homogeneous pre-stressed solid is considered. By use of a cubic elastic potential, thermoelasticity relations for bodies with inhomogeneous initial strains are obtained and equations describing the displacement dynamics are formulated. The coefficients of the equations are functions of the initial strain components. An iterative approach for solving boundary-value problems for the obtained system of equations with variable coefficients is developed. This approach enables to reduce problems concerning the determination of thermoelastic displacements in a body with inhomogeneous initial strains to a sequence of problems with fixed coefficients. Possibilities for the application of the developed approach and a mathematical model for the creation of new methods for a non-destructive determination of elastic-strain fields and residual stresses in solids are discussed.

Keywords Iterative methods · Inverse problems · Nondestructive testing · Pre-stressed bodies · Residual stresses · Thermoelastic disturbances

1 Introduction

The so-called fully and partly destructive methods for the determination of residual stresses in solids are well known [1; Chaps. 3, 4, 9]. Usually, for this purpose, dissection of the object into separate parts [1; pp. 98–102, 151–153]; drilling of through or deaf holes [2], removal of layers of material [1; pp. 60–74], formation of flutes [3] etc. are used. All this results in a release of part of the elastic energy accumulated in the body and in its deformation. By measuring displacements or strains on the body surface, one can assess the residual stresses acting in the body before the energy release. Obviously, the lesser destructive the method is, the smaller are the strains induced in the object and the more sensitive the instruments for measuring the surface displacements (or strains) should be.

Holographic speckle interferometry enables to measure the surface-displacement components and their spatial gradients with high resolution and precision. A modified version of this technique, which is known as electronic speckle interferometry or shearography [4], enables to register sequences of interferograms in digital format at a high

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rate, which can therefore be used to study non-stationary processes. Application of these high-precision measurement techniques enables to reduce substantially the requirements to the necessary level of surface deformations when one of the abovementioned methods of determining residual stresses is implemented [2].

In this connection a possibility to create a new nondestructive method for determining residual stresses on the basis of data obtained by measuring the irreversible deformation caused by local heating was studied in [5, 6]. Here the following idea has been worked out.

Thermal stresses, caused by local heating, being imposed on residual ones, can reach the elasticity limit of the material and result in plastic deformation. Hence, if the thermal stresses are known, then depending on the appearance of the plastic deformations, one can judge the level of the residual stresses which acted prior to the local heating.

Due to this method, the first holographic exposition should be done before local heating of the object in its homogeneous thermal state and the second after the temperature field of the object, disturbed by heating, has returned to its original homogeneous value. On the basis of the holographic interferograms, obtained in such a way, the displacement-vector components of the plastic deformation, caused by heating and cooling of the body, can be retrieved.

But it is worth pointing out that this method cannot be quite considered as nondestructive, since irreversible changes (plastic deformation) caused by heating still arise in the object. Therefore, alternative approaches, not resulting in irreversible changes of the object, are interesting to study.

Our purpose in this paper is to investigate an approach towards the creation of a nondestructive method for determining the stress–strain state of solids using data obtained by measuring parameters of the thermoelastic processes in the object.

The idea of the method [7] is that thermoelastic processes, initiated in a pre-stressed solid by external heating, can interact with the initial strain field. So, parameters of these processes (for instance, surface displacements or strains) have become dependent on the initial stress–strain state of the body. Measuring these parameters by use of the speckle holographic interferometry technique, one can obtain information that may be used to retrieve the initial stress–strain state of the body.

This method allows the parameters of the external heat influence to be chosen in a way that does not result in irreversible deformation. So the first holographic exposure in this case should be also done before starting heating the object, but the second one must be done during the heating–cooling process at a moment of time when the thermoelastic processes caused by external heating have not yet finished. The parameters of the thermoelastic processes can be measured at several time moments during the heating–cooling process. This enables to increase the body of information obtained in such a way.

In [8, 9] a variational approach to strain-fields retrieval was developed. This approach is based on three elements: a mathematical model of the initial stress–strain state, a set of informative parameters collected by sounding the object with an external field, and a model describing the interaction of the sounding field with the strained object. This model establishes relationships between the measured informative parameters and strain (and/or stress) distributions in the body.

To implement this approach for the case when an external heat flux is applied to probe the object, it is necessary to establish equations which connect the thermoelastic displacements (and/or strains) at the surface, which can be used as the informative parameters, with parameters of the initial stress–strain state of the body.

With this in mind, a mathematical model describing the thermoelastic behaviour of a piece-wise homogeneous pre-stressed solid is considered here. Starting from a cubic elastic potential, linearized thermoelasticity equations, the coefficients of which are functions of the components of the initial elastic strain tensor, will be defined and appropriate equations for the dynamics of the displacements given.

An iterative method for solving the boundary-value problems, formulated within the framework of the model, is developed. It allows reducing the problem with variable coefficients to some sequence of problems with constant coefficients.

Possibilities for applying the developed mathematical model to create nondestructive methods for determining residual stresses in solids, using data obtained by measuring thermoelastic displacements or strain components on the body surface during heating, will be discussed.

2 Mathematical model for initial stress–strain state

We consider a piece-wise homogeneous solid body $\mathcal{B} = \cup\mathcal{B}_\alpha$ in which, at a fixed homogeneous temperature T_0, stresses $\sigma = \{\sigma_{ij}\}$ act. In this state the body occupies an area $\mathcal{V}$ in three-dimensional Euclidean space. The arrangement of the material points $\mathcal{X} \in \mathcal{B}_\alpha$ of the body in the area $\mathcal{V}$ will be called its current configuration. The components σ_{ij} of the Cauchy stress tensor σ are continuously differentiable within the bounded areas $\mathcal{V}_\alpha$ of each part $\mathcal{B}_\alpha$; these are functions which satisfy in these areas the equilibrium equations

$$\frac{\partial \sigma_{ij}}{\partial x_i} + F_j = 0. \tag{1}$$

On the external body boundary $\partial\mathcal{V}$ and the surfaces $\mathcal{S}_\mu$ dividing different body parts $\mathcal{B}_\alpha$, the components σ_{ij} satisfy the conditions:

$$\sigma_{ij} n_j\big|_{\partial\mathcal{V}} = f_i, \quad \left[\sigma_{ij}\nu_j\right]_{\mathcal{S}_\mu} = 0. \tag{2}$$

Here $\mathbf{F} = \{F_i\}$ and $\mathbf{f} = \{f_i\}$ are the external volumetric and surface forces, respectively; x_i stands for the Cartesian coordinates of material points $\mathcal{X} \in \mathcal{B}_\alpha$ in the current configuration $\mathcal{V}$; n_j and ν_j denote the Cartesian components of the unit vectors normal to $\partial\mathcal{V}$ and $\mathcal{S}_\mu$; brackets $[...]_{\mathcal{S}_\mu}$ denote the first-kind discontinuity on the surface $\mathcal{S}_\mu$; use of repeated indices means that the sum over the entire index range has to be taken.

We require the strain components ε_{ij} to be small enough and purely elastic within the restricted areas $\mathcal{V}_\alpha$ of each body part $\mathcal{B}_\alpha$, so the distinction between these tensor components with respect to the coordinate systems of reference $\mathcal{V}_0$ and the current $\mathcal{V}$ configurations becomes negligible. Then the following relation applies:

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{3}$$

Here u_i stands for Cartesian components of the displacement vector $\mathbf{u} = \{u_i\}$ which determines the body transition from the reference non-stressed configuration $\mathcal{V}_0$ to the current one $\mathcal{V}$. The vector $\mathbf{u}$ is assumed to be a twice continuously differentiable vector function of coordinates x_i within the areas of each body part $\mathcal{B}_\alpha$, but on the surfaces $\mathcal{S}_\mu$ it can have discontinuous jumps:

$$[\mathbf{u}]_{\mathcal{S}_\mu} = \mathbf{u}_\mu. \tag{4}$$

Here $\mathbf{u}_\mu$ denotes some vector function defined on the surface $\mathcal{S}_\mu$.

This means that the initial strains are compatible within the all body parts $\mathcal{B}_\alpha$ but in null sets of points, i.e., on the interface surfaces $\mathcal{S}_\mu$ deformation incompatibility can exist. In the model conditions (4) are responsible for residual stresses acting in the body in the current configuration. The only origin of the residual stresses is the deformation incompatibility occurring on the boundaries $\mathcal{S}_\mu$. The jump vector function $\mathbf{u}_\mu(\mathbf{r})$, $\mathbf{r} \in \mathbf{S}_\mu$ is a measure of this incompatibility on the surface $\mathbf{S}_\mu$. If a vector $\mathbf{u}_\mu(\mathbf{r})$, $\mathbf{r} \in \mathbf{S}_\mu$ is zero-valued, then the initial strain components are compatible on the corresponding surface.

Besides the residual, stresses caused by external loading (by surface $\mathbf{f} = \{f_i\}$ and volumetric $\mathbf{F} = \{F_i\}$ forces) act in the body. We assume the stresses caused by external loading are small enough, so they cannot change the residual ones. This means, in particular, that the functions $\mathbf{u}_\mu(\mathbf{r})$, $\mathbf{r} \in \mathbf{S}_\mu$ do not depend on the body's strain–stress state.

Under such conditions each part $\mathcal{B}_\alpha$ is a hyper-elastic body for which an elastic potential $\Phi = \Phi_\alpha(\varepsilon_{ij})$ exists. Then, one-to-one relations between the components of the stress σ_{ij} and strain ε_{ij} tensors in the volume $\mathcal{V}_\alpha$ of each body part $\mathcal{B}_\alpha$,

$$\varepsilon_{ij} = \varepsilon_{ij}(\sigma_{kl}), \quad \sigma_{ij} = \sigma_{ij}(\varepsilon_{kl}), \tag{5}$$

can be expressed by

$$\sigma_{ij} = \frac{1}{2}\left(\frac{\partial \Phi}{\partial \varepsilon_{ij}} + \frac{\partial \Phi}{\partial \varepsilon_{ji}}\right). \tag{6}$$

If we chose the cubic elastic potential as follows:

$$\Phi_0(\varepsilon_{ij}) = \frac{1}{2} C_{ijkl}\,\varepsilon_{ij}\,\varepsilon_{kl} + \frac{1}{6}\Gamma_{ijklmn}\,\varepsilon_{ij}\,\varepsilon_{kl}\,\varepsilon_{mn}, \tag{7}$$

the elasticity relations (5) will look like

$$\sigma_{ij} = \left(C_{ijkl} + \frac{1}{2}\Gamma_{ijklmn}\,\varepsilon_{mn}\right)\varepsilon_{kl}. \tag{8}$$

Here C_{ijkl} and Γ_{ijklmn} $(i, j, k, l, m, n = 1, 2, 3)$ denote elasticity coefficients of the second and the third orders—components of elastic tensors of rank four and six correspondingly. The tensors C_{ijkl} and Γ_{ijklmn} are piecewise-homogeneous in the body volume $\mathcal{V}$, symmetric with respect to each pair of indices and invariant with respect to permutations of these pairs.

For isotropic solids with the Murnaghan elastic potential, the elasticity tensors can be expressed as

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\,(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \quad \Gamma_{ijklmn} = \Gamma'_{((ij)(kl)(mn))}, \tag{9}$$

$$\Gamma'_{ijklmn} = 2\,(l + 2m)\,\delta_{ij}\,\delta_{kl}\delta_{mn} - 6m\delta_{ij} \in_{pkm}\in_{pln} + n \in_{ikm}\in_{jln}. \tag{10}$$

Here δ_{ij} denotes Kronecker's delta, $\in_{ikm}$ stands for Levi-Civita symbols; $\lambda,\ \mu$ and $l,\ m,\ n$ stand for the second (Lamé constants) and third (Murnaghan coefficients) order elastic modulus, which are treated as piecewise-homogeneous parameters in the body volume $\mathcal{V}$. Parenthesis in the notation $\Gamma'_{((ij)(kl)(mn))}$ denote symmetrization of the components with respect to the corresponding groups of indices.

Equations (1), (3), (8), together with the boundary-interface conditions (2) and (4), form a closed mathematical model for the stress–strain state of the heterogeneous body, on the interface boundaries $\mathcal{S}_\mu$ of which the deformation incompatibility, resulting in residual stresses, can occur. This means that, if the vector functions $\mathbf{f}\,(\mathbf{r})$, $\mathbf{r} \in \partial\mathcal{V}$; $\mathbf{F}\,(\mathbf{r})$, $\mathbf{r} \in \mathcal{V}_\alpha$ and $\mathbf{u}_\mu\,(\mathbf{r})$, $\mathbf{r} \in \mathcal{S}_\mu$ are given, then correct direct problems for determining the stress–strain state can be formulated within the framework of the model.

When elastic bodies, such as metals, glass, ceramics etc, are considered, there is no need to use the nonlinear equations (8) in direct problems. For such objects the linearized elasticity relations,

$$\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl}, \tag{11}$$

are sufficient to use instead of (8), when direct problems are defined. This is possible because for such objects the elastic strains are much less than unity and the elastic moduli C_{ijkl} and Γ_{ijklmn} are of the same order of magnitude.

Such a simplification enables us to reduce the mathematical model of the initial stress–strain state to linear equations in the displacements, which for isotropic bodies can be written, for instance, in the form

$$\mu\nabla^2 u_i + (\lambda + \mu)\,\frac{\partial^2 u_k}{\partial x_i \partial x_k} + F_i = 0, \tag{12}$$

where ∇^2 stands for 3-D Laplace operator.

In this case the conditions (2) can also be reduced to linear relations for the displacements:

$$C_{ijkl}\frac{\partial u_k}{\partial x_l}\bigg|_{\partial\mathcal{V}} n_j = f_i, \quad \left[C_{ijkl}\frac{\partial u_k}{\partial x_l}\right]_{\mathbf{S}_\mu} \nu_j = 0. \tag{13}$$

The linear direct problem (12), (13), (4) can be divided into two independent ones; these are the problems for the determination of the residual and mechanical (causing by external loading) stresses. To determine the stresses caused by external loading, the system (12) should be solved subject to the conditions (4), in which $\mathbf{u}_\mu \equiv 0$, and the conditions (13). To find the residual stresses, the system (12), in which $F_i \equiv 0$, should be solved subject to the conditions (13), in which $f_i \equiv 0$, and conditions (4).

If the functions $\mathbf{u}_\mu\,(\mathbf{r})$, $\mathbf{r} \in \mathcal{S}_\mu$ and/or $\mathbf{f}\,(\mathbf{r})$, $\mathbf{r} \in \partial\mathcal{V}$; $\mathbf{F}\,(\mathbf{r})$, $\mathbf{r} \in \mathcal{V}_\alpha$ are unknown a priori, it becomes impossible to formulate correct direct problems for the determination of the stress–strain state in the body. In this case one might consider inverse problems.

In the next chapters the nonlinear elasticity relations (8) will be used to construct a thermoelasticity model for prestressed bodies, in which residual stresses, caused by deformation incompatibility occurring on some inside surfaces, and stresses, caused by external loading, act.

3 Thermoelasticity relations for pre-stressed solids

Let a thermoelastic disturbance, caused by external heating of the body, be imposed on the initial stress–strain state. Variations of the temperature θ are small enough; therefore the total stresses, which include the mechanical (caused by external loading), residual and thermal components, cannot cause irreversible changes in the material. This means that additional residual stresses do not occurr and the disturbance strains e_{ij} are small, including the elastic e^e_{ij} and temperature $\alpha_{ij}\theta$ terms:

$$e_{ij} = e^e_{ij} + \alpha_{ij}\theta \tag{14}$$

and are compatible in the areas $\mathcal{V}_\alpha$ of each body part $\mathcal{B}_\alpha$:

$$e_{ij} = \frac{1}{2}\left(\frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i}\right). \tag{15}$$

Here α_{ij} denotes the components of the thermal-expansion tensor; w_i stands for components of the displacement vector $\mathbf{w}$ from the current configuration $\mathcal{V}$ to the disturbed one $\tilde{\mathcal{V}}$ which is a twice continuously differentiable vector-function in the areas $\mathcal{V}_\alpha$ of each body part $\mathcal{B}_\alpha$ and continuous on the boundary $\mathcal{S}_\mu$ dividing these parts:

$$[\mathbf{w}]_{\mathcal{S}_\mu} = 0. \tag{16}$$

The geometrically linear approach is used, allowing the strain components for the disturbed state $\tilde{\varepsilon}_{ij}$ to be presented as the sum $\tilde{\varepsilon}_{ij} = \varepsilon_{ij} + e_{ij}$, and in this state the elastic potential $\tilde{\Phi}$ can be taken as

$$\tilde{\Phi} = \Phi_0\left(\varepsilon_{ij} + e_{ij} - \alpha_{ij}\theta\right). \tag{17}$$

Let us expand the function in the right-hand side of (13) into a power series with respect to the variables e_{ij} and $\alpha_{ij}\theta$ and, taking into account that they are small, retain terms not higher than order two. Then, for a potential such as (7), we obtain:

$$\tilde{\Phi} = \Phi_0\left(\varepsilon_{ij}\right) + \sigma_{ij}\left(e_{ij} - \alpha_{ij}\theta\right) + \frac{1}{2}\left(C_{ijkl} + \Gamma_{ijklmn}\,\varepsilon_{mn}\right)\left(e_{ij} - \alpha_{ij}\theta\right)\left(e_{kl} - \alpha_{kl}\theta\right). \tag{18}$$

For the disturbed state, determined by the parameters $\left(\tilde{\sigma}_{ij}, \tilde{\varepsilon}_{ij}\right)$, Eq. (6) should be written as

$$\tilde{\sigma}_{ij} = \frac{1}{2}\left(\frac{\partial\tilde{\Phi}}{\partial\tilde{\varepsilon}_{ij}} + \frac{\partial\tilde{\Phi}}{\partial\tilde{\varepsilon}_{ji}}\right). \tag{19}$$

Substituting the relation (18) in (19) , we come to the following linearized thermoelasticity relations for pre-stressed solids:

$$s_{ij} = \left(C_{ijkl} + \Gamma_{ijklmn}\,\varepsilon_{mn}\right)e_{kl} - \left(C_{ijkl}\,\alpha_{kl} + \Gamma_{ijklmn}\,\alpha_{kl}\,\varepsilon_{mn}\right)\theta. \tag{20}$$

Here $s_{ij} = \tilde{\sigma}_{ij} - \sigma_{ij}$ stands for the components of the additional stresses caused by heating.

After solving the set of equations (15) with respect to the components ε_{ij} and taking into account that the components ε_{ij} are small, we obtain:

$$e_{ij} = \left(S_{ijkl} - G_{ijklmn}\,\varepsilon_{mn}\right)s_{kl} + \alpha_{ij}\theta, \tag{21}$$

where

$$G_{ijklmn} = S_{ijop}S_{klsg}\Gamma_{opsgmn};$$

here S_{ijkl} stands for components of the elastic compliance tensor:

$$S_{ijkl} = \frac{\lambda}{2\mu\left(3\lambda + 2\mu\right)}\delta_{ij}\delta_{kl} + \frac{1}{\mu}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right). \tag{22}$$

We will take into account the dependence of the coefficients of thermal expansion on the initial strains as

$$\alpha_{ij} = \alpha^0_{ij} + A_{ijkl}\varepsilon_{kl}. \tag{23}$$

Here α_{ij}^0 stands for components of the thermal-expansion tensor of the body in its unstrained state, A_{ijkl} are the components of a material tensor of rank four that take into account the influence of the intial strain of the body on its thermal expansion.

For an isotropic body we have

$$\alpha_{ij}^0 = \alpha\delta_{ij}, \quad A_{ijkl} = \alpha_1\delta_{ij}\delta_{kl} + \alpha_2(\delta_{ik}\delta_{jl} + \delta_{il}\delta j_k). \tag{24}$$

Here α is the thermal-expansion coefficient of the isotropic body, α_1 and α_2 denote material coefficients that account for the influence of the strain tensor on thermal expansion.

If the terms of order three, $\alpha_1\theta\varepsilon_{ij}\varepsilon_{kl}$ and $\alpha_2\theta\varepsilon_{ij}\varepsilon_{kl}$, are disregarded, Eq. (20) can be written as

$$s_{ij} = \left(C_{ijkl} + \Gamma_{ijklmn}\,\varepsilon_{mn}\right) e_{kl} - \left(\beta_{ij} + L_{ijkl}\,\varepsilon_{kl}\right)\theta. \tag{25}$$

Here

$$\begin{aligned} &\beta_{ij} = \beta\delta_{ij},\ \ L_{ijkl} = \beta_1\delta_{ij}\delta_{kl} + \beta_2\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right),\ \ \beta = (3\lambda + 2\mu)\,\alpha, \\ &\beta_1 = (6l + n)\,\alpha + (3\lambda + 2\mu)\,\alpha_1 + 2\lambda\alpha_2,\ \ \beta_2 = -\alpha n + 4\mu\alpha_2. \end{aligned} \tag{26}$$

If the initial strains vanish or when the elastic and temperature-expansion properties of the body do not depend on strain, then Eq. (25) turns into the known thermoelasticity relations for isotropic bodies.

4 Equations of motion for disturbances

Considering that the body is in equilibrium initially, we have the following equations describing the dynamics of the disturbed state:

$$\rho\frac{\partial^2 w_i}{\partial t^2} = \frac{\partial s_{ij}}{\partial x_j}, \tag{27}$$

where ρ is the mass density, and t the time variable.

Starting from here and taking into account Eqs. 11 and 15, we obtain the following linearized thermoelastic equations for the displacements:

$$\rho\frac{\partial^2 w_i}{\partial t^2} = \frac{\partial}{\partial x_l}\left(\left(C_{ijkl} + \Gamma_{ijklmn}\,\varepsilon_{mn}\right)\frac{\partial w_k}{\partial x_j}\right) - \frac{\partial}{\partial x_j}\left(\left(\beta_{ij} + L_{ijkl}\,\varepsilon_{kl}\right)\theta\right), \tag{28}$$

which are valid for each homogeneous part $\mathcal{B}_\alpha$. On the external, $\partial\mathcal{V}$, and internal, $\mathcal{S}_\mu$, surfaces the conditions

$$\begin{aligned} &\left(C_{ijkl} + \Gamma_{ijklmn}\,\varepsilon_{mn}\right)\left(\frac{\partial w_k}{\partial x_l} - \alpha_{kl}\theta\right) n_j\bigg|_{\partial\mathcal{V}} = 0, \\ &\left[\left(C_{ijkl} + \Gamma_{ijklmn}\,\varepsilon_{mn}\right)\left(\frac{\partial w_k}{\partial x_l} - \alpha_{kl}\theta\right)\right]\nu_j\bigg|_{\mathcal{S}_\lambda} = 0,\quad [w_i]_{\mathcal{S}_\lambda} = 0 \end{aligned} \tag{29}$$

hold.

Inasmuch as the initial strains ε_{ij} are spatially inhomogeneous, the coefficients of the system (28) and the relations (29) are dependent on the spatial coordinates.

5 Iterative process

Let us rewrite Eq. 28 and conditions (29) in the form

$$\rho\frac{\partial^2 w_i}{\partial t^2} = C_{ijkl}\frac{\partial^2 w_k}{\partial x_j\partial x_l} - \frac{\partial}{\partial x_j}\left(\left(\beta_{ij} + L_{ijkl}\,\varepsilon_{kl}\right)\theta\right) + \frac{\partial}{\partial x_j}\left(\Gamma_{ijklmn}\,\varepsilon_{mn}\frac{\partial w_k}{\partial x_l}\right), \tag{30}$$

$$\begin{aligned} &C_{ijkl}\frac{\partial w_k}{\partial x_l}n_j\bigg|_{\partial\mathcal{V}} = \left(\beta_{ij} + L_{ijkl}\,\varepsilon_{kl}\right)\theta n_j\big|_{\partial\mathcal{V}} - \Gamma_{ijklmn}\,\varepsilon_{mn}\frac{\partial w_k}{\partial x_l}n_j\bigg|_{\partial\mathcal{V}}, \\ &\left[C_{ijkl}\frac{\partial w_k}{\partial x_l}\nu_j\right]_{\mathbf{S}_\lambda} = \left[\left(\beta_{ij} + L_{ijkl}\,\varepsilon_{kl}\right)\theta\nu_j\right]_{\mathcal{S}_\lambda} - \left[\Gamma_{ijklmn}\,\varepsilon_{mn}\tfrac{\partial w_k}{\partial x_l}\nu_j\right]_{\mathcal{S}_\lambda}, \end{aligned} \tag{31}$$

and compare the terms in the right-hand sides of (30) and the relations (31).

The elastic constants C_{ijkl} and Γ_{ijklmn} for the material of interest have values of the same order; let us denote this order by C

$$\|C_{ijkl}\| \sim \|\Gamma_{ijklmn}\| \sim C. \tag{32}$$

Let us introduce norms for the tensor fields ε_{ij}, e_{ij} and their gradients $\partial\varepsilon_{ij}/\partial x_k$, $\partial e_{ij}/\partial x_k$:

$$\|\varepsilon_{ij}\| = \mathcal{E}, \quad \|e_{ij}\| = E, \quad \left\|\frac{\partial\varepsilon_{ij}}{\partial x_k}\right\| = D_\varepsilon, \quad \left\|\frac{\partial e_{ij}}{\partial x_k}\right\| = D_e. \tag{33}$$

Then the first term in the right-hand side of Eq. 30 can be evaluated as CD_e and the last one as $C\mathcal{E}D_e + CDd_\varepsilon$. From this follows that the last term in the right-hand side of (30) will be small compared with the first one if the following condition is satisfied:

$$\mathcal{E} + E\frac{D_\varepsilon}{D_e} \ll 1. \tag{34}$$

In this model an elastic initial state and small thermoelastic disturbances are considered, so for metals we have

$$E < \mathcal{E} \ll 1. \tag{35}$$

This means that condition (34) will be satisfied if the order of magnitude of the ratio D_ε/D_e does not exceed unity significantly

$$\frac{D_\varepsilon}{D_e} \leq 1. \tag{36}$$

Hence we may conclude the following: the smaller the gradient of the initial strains and the higher the strain gradients of the disturbed state are, the better condition (34) will be satisfied.

Similarly, we can see that from the validity of inequality (35) follows that the last terms in the right-hand side of the first and second equations of (36) are small compared to the left-hand sides of these relations, respectively.

Thus, when the restrictions (35), (36) on the initial and disturbed strain fields are satisfied, the problem (30), (31) can be solved by an iterative method.

Due to this, if the solution of the problem

$$\rho\frac{\partial^2 w_i}{\partial t^2} = C_{ijkl}\frac{\partial^2 w_k}{\partial x_j \partial x_l} - \frac{\partial}{\partial x_j}\left(\left(\beta_{ij} + L_{ijkl}\,\varepsilon_{kl}\right)\theta\right), \tag{37}$$

$$\begin{aligned} &C_{ijkl}\tfrac{\partial w_k}{\partial x_l}n_j\Big|_{\partial\mathcal{V}} = \left(\beta_{ij} + L_{ijkl}\,\varepsilon_{kl}\right)\theta n_j\big|_{\partial\mathcal{V}}, \\ &\left[C_{ijkl}\tfrac{\partial w_k}{\partial x_l}\nu_j\right]_{\mathcal{S}_\lambda} = \left[\left(\beta_{ij} + L_{ijkl}\,\varepsilon_{kl}\right)\theta\nu_j\right]_{\mathcal{S}_\lambda}, \quad [w_i]_{\mathcal{S}_\lambda} = 0, \end{aligned} \tag{38}$$

is taken as a first approximation $v_i^{(0)} = w_i$, then each next $(p+1)$-approximation $(p = 0, 1, \ldots)$ will be obtained as a solution of the following problem

$$\rho\frac{\partial^2 v_i^{(p+1)}}{\partial t^2} = C_{ijkl}\frac{\partial^2 v_k^{(p+1)}}{\partial x_j \partial x_l} + \frac{\partial}{\partial x_j}\left(\Gamma_{ijklmn}\,\varepsilon_{mn}\frac{\partial w_k^{(p)}}{\partial x_l}\right), \tag{39}$$

$$\begin{aligned} &C_{ijkl}\frac{\partial v_k^{(p+1)}}{\partial x_l}n_j\bigg|_{\partial\mathcal{V}} = -\,\Gamma_{ijklmn}\,\varepsilon_{mn}\frac{\partial w_k^{(p)}}{\partial x_l}n_j\bigg|_{\partial\mathcal{V}}, \\ &\left[C_{ijkl}\frac{\partial v_k^{(p+1)}}{\partial x_l}\nu_j\right]_{\mathcal{S}_\lambda} = -\left[\Gamma_{ijklmn}\,\varepsilon_{mn}\frac{\partial w_k^{(p)}}{\partial x_l}\nu_j\right]_{\mathcal{S}_\lambda}, \quad \left[v_i^{(p+1)}\right]_{\mathcal{S}_\lambda} = 0. \end{aligned} \tag{40}$$

Here $v_i^{(p)}$ stands for the improvement of the displacements after the p-th iteration: $v_i^{(p)} \equiv w_k^{(p)} - v_k^{(0)}$; the upper index denotes the iteration number.

We can arrive at another iterative process by representing the relations (15), (25), (27) in the form

$$\begin{aligned} e_{ij}^{(0)} &= \frac{1}{2}\left(\frac{\partial w_i^{(0)}}{\partial x_j} + \frac{\partial w_j^{(0)}}{\partial x_i}\right), \quad \rho \frac{\partial^2 w_i^{(0)}}{\partial t^2} = \frac{\partial s_{ij}^{(0)}}{\partial x_j}, \quad s_{ij}^{(0)} = C_{ijkl} e_{kl}^{(0)} - \left(C_{ijkl} + \Gamma_{ijklmn}\,\varepsilon_{mn}\right)\alpha_{kl}\theta, \\ e_{ij}^{(p+1)} &= \frac{1}{2}\left(\frac{\partial v_i^{(p+1)}}{\partial x_j} + \frac{\partial v_j^{(p+1)}}{\partial x_i}\right), \quad \rho \frac{\partial^2 v_i^{(p+1)}}{\partial t^2} = \frac{\partial p_{ij}^{(p+1)}}{\partial x_j}, \quad p_{ij}^{(p+1)} = C_{ijkl} u_{kl}^{(p+1)} + \Gamma_{ijklmn}\,\varepsilon_{mn} e_{kl}^{(p)}, \end{aligned} \tag{41}$$

where $p_{ij}^{(p+1)}$ and $u_{ij}^{(p+1)}$ are iterative improvements of the perturbed stress and strain tensors:

$$p_{ij}^{(p+1)} = s_{ij}^{(p+1)} - s_{ij}^{(0)}, \quad u_{ij}^{(p+1)} = e_{ij}^{(p+1)} - e_{ij}^{(0)}, \quad p = 0, 1, 2, \ldots \tag{42}$$

From the first and fourth equation of (41) we have that the strain field is compatible during each iteration:

$$\in_{ikl}\in_{jmn} \frac{\partial^2 e_{km}^{(0)}}{\partial x_l \partial x_n} = 0, \quad \in_{ikl}\in_{jmn} \frac{\partial^2 u_{km}^{(p+1)}}{\partial x_l \partial x_n} = 0. \tag{43}$$

Starting from (41) and (43), we may construct an iterative sequence for the stresses that is equivalent to (37)–(40).

Thus, if the initial strains are known, the perturbed thermoelastic state can be determined by sequentially solving boundary-value problems of type (37)–(40). The equations and boundary conditions in these problems for different iterations differ from one another only by their right-hand sides. These right-hand terms can be calculated during each step from the solution obtained during the previous iteration.

6 Case of 2-D initial stress–strain state

Let the initial strain state be the plane one:

$$\varepsilon_{ij} = \varepsilon_{ij}(x_1, x_2), \quad i, j = 1, 2, \quad \varepsilon_{i3} = \varepsilon_{3i} = 0, \quad \varepsilon_{33} = \text{const.} \tag{44}$$

Let the temperature disturbance also be 2-D: $\theta = \theta(x_1, x_2, t)$. Then the displacement vector can be taken in the form

$$w = (w_1(x_1, x_2), \quad w_2(x_1, x_2), e_{33}x_3)^T. \tag{45}$$

Here e_{33} is a parameter independent of the x_1, x_2-coordinates, the value of which is dependent on the clamping condition acting at infinity in the direction x_3.

In the quasi-static case (37) and (39) can be transformed into:

$$\mu \Delta v_i^{(p)} + (\lambda + \mu)\frac{\partial}{\partial x_i}\left(\frac{\partial v_1^{(p)}}{\partial x_1} + \frac{\partial v_2^{(p)}}{\partial x_2}\right) = \frac{\partial f_{ij}^{(p)}(\varepsilon_{mn})}{\partial x_j}, \quad i, j = 1, 2. \tag{46}$$

Here Δ stands for the Laplace operator in Cartesian coordinates in the plane x_1, x_2, and the following notations for the functions $f_{ij}^{(p)}(\varepsilon_{mn})$ for different iterations are used:

$$f_{ij}^{(0)} = \theta\left(\left(\beta + (\beta_1 + \beta_2)\,\varepsilon\right)\delta_{ij} + \beta_2\left(\varepsilon_{ij} - \delta_{ij}\varepsilon\right)\right), \quad i, j = 1, 2 \tag{47}$$

for $p = 0$, and

$$\begin{aligned} f_{11}^{(p)} &= 6m \sum_{i,j=1,2} \varepsilon_{ij} e_{ij}^{(p-1)} + 2(l - m)\ \varepsilon e^{(p-1)} + n\left(\varepsilon_{33} e_{22}^{(p-1)} + \varepsilon_{22} e_{33}^{(p-1)}\right), \\ f_{12}^{(p)} &= f_{21}^{(p)} = -n\left(\varepsilon_{12} e_{33}^{(p-1)} + \varepsilon_{33} e_{12}^{(p-1)}\right), \\ f_{22}^{(p)} &= 6m \sum_{i,j=1,2} \varepsilon_{ij} e_{ij}^{(p-1)} + 2(l - m)\,\varepsilon e^{(p-1)} + n\left(\varepsilon_{33} e_{11}^{(p-1)} + \varepsilon_{11} e_{33}^{(p-1)}\right), \end{aligned} \tag{48}$$

for $p \geq 1$.

Here

$$\varepsilon = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}, \quad e^{(p)} = e_{11}^{(p)} + e_{22}^{(p)} + e_{33}^{(p)}. \tag{49}$$

Thus, in the 2-D quasi-static case, the iterative process is completed at each step of the system of equations

$$\begin{aligned} \mu \Delta v_1 + (\lambda + \mu) \frac{\partial}{\partial x_1} \left(\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} \right) &= \frac{\partial f_{11}}{\partial x_1} + \frac{\partial f_{12}}{\partial x_2}, \\ \mu \Delta v_2 + (\lambda + \mu) \frac{\partial}{\partial x_2} \left(\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} \right) &= \frac{\partial f_{21}}{\partial x_1} + \frac{\partial f_{22}}{\partial x_2} \end{aligned} \tag{50}$$

in which $f_{11}, f_{12} = f_{21}, f_{22}$ are given functions.

Let us introduce the following functions

$$\eta \equiv \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2}, \quad \omega \equiv \frac{1}{2} \left(\frac{\partial v_1}{\partial x_2} - \frac{\partial v_2}{\partial x_1} \right), \tag{51}$$

which, as follows from (50), satisfy the equations

$$\Delta \eta = (\lambda + 2\mu)^{-1} f_\eta, \quad \Delta \omega = (2\mu)^{-1} f_\omega, \tag{52,53}$$

where

$$f_\eta \equiv \frac{\partial^2 f_{11}}{\partial x_1^2} + 2 \frac{\partial^2 f_{12}}{\partial x_1 \partial x_2} + \frac{\partial^2 f_{22}}{\partial x_2^2}, \quad f_\omega \equiv \frac{\partial^2}{\partial x_1 \partial x_2} (f_{11} - f_{22}) - \left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} \right) f_{12}. \tag{54,55}$$

The parameters η and ω determine variations of the elementary coordinate area and rotation angle of an elementary linear element in the plane $x_1 x_2$, respectively. If these functions are known, the displacement components v_1, v_2 can be determined by solving the equations

$$\Delta v_1 = \frac{\partial \eta}{\partial x_1} + 2 \frac{\partial \omega}{\partial x_2}, \quad \Delta v_2 = \frac{\partial \eta}{\partial x_2} - 2 \frac{\partial \omega}{\partial x_1} \tag{56}$$

which follow from (51).

To solve the initial-boundary-value problem for (50), it is sufficient, during each iteration, to choose some particular solutions $\eta = \eta(x_1, x_2)$, $\omega = \omega(x_1, x_2)$ of (52) and (53), then find the general solutions of (56) and subject these to the conditions (38) and (40).

Particular solutions of Eqs. (52) and (53) can be found by utilizing the elementary solution of the 2-D Laplace equation:

$$G(x_1, x_2, \xi_1, \xi_2) = \log \left((x_1 - \xi_1)^2 + (x_2 - \xi_2)^2 \right)^{-1/2}. \tag{57}$$

Then we will have

$$\begin{aligned} \eta(x_1, x_2) &= \frac{1}{\lambda + 2\mu} \int_0^\infty \int_0^\infty f_y(\xi_1, \xi_2) G(x_1, x_2, \xi_1, \xi_2) \, d\xi_1 d\xi_2, \\ \omega(x_1, x_2) &= \frac{1}{2\mu} \int_0^\infty \int_0^\infty f_\omega(\xi_1, \xi_2) G(x_1, x_2, \xi_1, \xi_2) \, d\xi_1 d\xi_2. \end{aligned} \tag{58}$$

Once the functions $\eta(x_1, x_2)$ and $\omega(x_1, x_2)$, are known, one can determine the right-hand sides of (56) and, using the elementary solutions (57), find particular solutions of these equations. To establish the solutions of the corresponding boundary-value problems for the Laplace equation, which satisfy conditions (38) and (40), potential theory, boundary elements or finite-element methods can be used.

7 Case of local temperature disturbance

If the temperature disturbance is local, the thermoelastic state caused by this disturbance will also be local. Hence, heating the object in an area sufficiently remote from the body surface and the boundaries, dividing its different

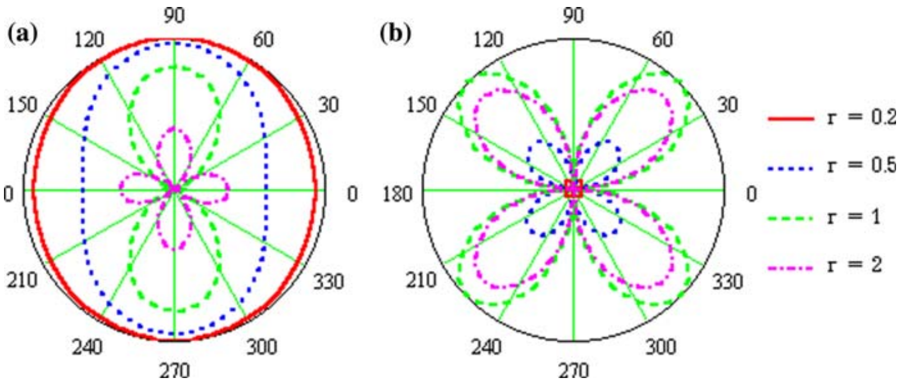

Fig. 1 η (**a**) and ω (**b**) versus the polar angle φ for different distances from the heating spot center

parts, one can use the particular solutions of the inhomogeneous equations (52), (53), for which the displacements vanish when the x_1, x_2-coordinates tend to infinity. Considering this case, let us study the thermoelastic deformation resulting from the axisymmetric temperature disturbance:

$$\theta = \theta(r), \quad r \equiv \sqrt{x_1^2 + x_2^2}, \tag{59}$$

where r stands for the polar radius.

Notice that the thermoelastic strain can be divided into an isotropic (uniform expansion) and an anisotropic part. The solution of (52) and (53) for the isotropic part can be found easily:

$$\eta(r) = (\lambda + 2\mu)^{-1} (\beta + \beta_1 \varepsilon) \theta(r), \quad \omega(r) = 0. \tag{60}$$

This solution contains two terms for η the first, $(\lambda + 2\mu)^{-1} \beta\theta(r)$, corresponds to thermal expansion of the body without initial strains and the second, $(\lambda + 2\mu)^{-1} \beta_1\varepsilon\theta(r)$, determines the influence of the isotropic part of the strain tensor on thermal expansion.

The solution for the anisotropic part can be found from (58). Taking into account the axial symmetry of the temperature field (59), one obtains:

$$\begin{aligned}\eta(r, \varphi, t) &= \frac{2\beta_2}{\lambda + 2\mu} \int_0^{2\pi} \int_0^{\infty} (\varepsilon_1 - \varepsilon_2) \cos 2\varphi' \frac{\partial\theta}{\partial r'} G\left(r, \varphi, r', \varphi'\right) dr' d\varphi' \\ &+ \frac{2\beta_2}{\lambda + 2\mu} \int_0^{2\pi} \int_0^{\infty} \left(\varepsilon_1 \sin^2 \varphi' + \varepsilon_2 \cos^2 \varphi'\right) \frac{\partial}{\partial r'} r' \frac{\partial\theta}{\partial r'} G\left(r, \varphi, r', \varphi'\right) dr' d\varphi',\end{aligned} \tag{61}$$

$$\omega(r, \varphi, t) = \frac{\beta_2}{2\mu} \int_0^{2\pi} \int_0^{\infty} (\varepsilon_1 - \varepsilon_2) \sin 2\varphi' \left(-\frac{\partial\theta}{\partial r'} + r' \frac{\partial^2\theta}{\partial r'^2}\right) G\left(r, \varphi, r', \varphi'\right) dr' d\varphi'. \tag{62}$$

Here

$$G\left(r, \varphi, r', \varphi'\right) = \log\left(r^2 + r'^2 - 2rr' \cos\left(\varphi - \varphi'\right)\right)^{-1/2},$$

where ε_1 and ε_2 are the main values of the initial strain tensor in the Cartesian coordinate system with origin in the heating-spot center. The polar angles φ and φ' are counted from the first main direction of the initial strain tensor.

If the temperature disturbance is applied in a sufficiently small area and the initial strain gradients are small enough, the components $\varepsilon_1, \ \varepsilon_2$ in (38) can be treated as independent spatial coordinates. In Fig. 1(a,b) the normalized angle dependencies of the parameters η (anisotropic components) and ω for $\varepsilon_2/\varepsilon_1 = -4$ and different distances r from the center of the temperature spot are presented.

8 Case of homogeneous heating

As we can see from (28), the non-homogeneous thermal stresses and strains will arise in a pre-stressed body, even under homogeneous external heating. This is caused by the dependence of the elastic moduli and the thermal-expansion constants on the initial stress–strain state. In this case the ratio D_ε/D_e is of order unity:

$$\frac{D_\varepsilon}{D_e} \cong 1. \tag{63}$$

Because of the smallness of the initial $\mathcal{E}$ and thermoelastic E strain-tensor norms, the inequality (34) is still valid. So the iterative process can be used in this case too. As a zero-order approximation, we have the following problem for this case:

$$\rho\frac{\partial^2 w_i}{\partial t^2} = C_{ijkl}\frac{\partial^2 w_k}{\partial x_j \partial x_l} - (T - T_0)\frac{\partial}{\partial x_j}\left(\beta_{ij} + L_{ijkl}\,\varepsilon_{kl}\right), \tag{64}$$

$$\begin{aligned} &C_{ijkl}\frac{\partial w_k}{\partial x_l} n_j\Big|_{\partial\mathcal{V}} = (T - T_0)\left(\beta_{ij} + L_{ijkl}\,\varepsilon_{kl}\right) n_j\big|_{\partial\mathcal{V}}, \\ &\left[C_{ijkl}\frac{\partial w_k}{\partial x_l}\nu_j\right]_{\mathcal{S}_\lambda} = (T - T_0)\left[\left(\beta_{ij} + L_{ijkl}\,\varepsilon_{kl}\right)\nu_j\right]\big|_{\mathcal{S}_\lambda}, \quad [w_i]_{\mathcal{S}_\lambda} = 0, \end{aligned} \tag{65}$$

where T_0 and T stand for homogeneous initial and current temperature of the body.

In the case of 2-D initial strain fields and a quasi-static approach, the functions f_η and f_ω in the right-hand sides of (52) and (53) for the zeroth approximation are

$$f_\eta = (T - T_0)\,(\beta_1 + \beta_2)\,\Delta\varepsilon, \quad f_\omega = (T - T_0)\,\beta_2\Delta\Omega, \tag{66}$$

where

$$\Omega = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right); \tag{67}$$

here u_1 and u_2 denote the displacement-vector components of the initial 2-D strain field.

We can now write down the next general solutions for the equations (52) and (53)

$$\eta = (T - T_0)\,\frac{(\beta_1 + \beta_2)\,(\varepsilon_{11} + \varepsilon_{22})}{\lambda + 2\mu} + H_\eta, \quad \omega = (T - T_0)\,\frac{\beta_2\Omega}{2\mu} + H_\omega, \tag{68,69}$$

where H_η and H_ω are piecewise-continuous harmonic functions. Choosing these functions, one can subject the solutions (68) and (69) to the conditions (65).

9 The inverse-problem statement

The mathematical model developed here enables to calculate the thermoelastic strains and displacements that are caused by small temperature disturbances in bodies with elastic initial deformations distributions when the initial strain-tensor components $\varepsilon_{ij}(x_1, x_2, x_3)$ and temperature field $\theta(x_1, x_2, x_3, t)$ are given.

On the other hand, if a known heat flux acts upon the body and the components of the thermoelastic displacements $w_i(x_1, x_2, x_3, t)$ and/or strain $e_{ij}(x_1, x_2, x_3, t)$ are measured on some sub-area of the body (e.g. its surface), we obtain data that can be used simultaneously with this model to formulate inverse problems for nondestructive determinations of the initial stress–strain state.

In problems concerning the tomographical reconstruction of tensor fields in solids based on data obtained by sounding the object by an external beam, it is often suitable to use integral relations that connect the measured informative parameters of the probing field with parameters of the stress–strain state being reconstructed. Equations (61) and (62) give an example of such relations that express the parameters η and ω of the disturbed thermoelastic state in terms of the initial strain components ε_{ij} and temperature θ distributions in the body volume. Thus, measuring the parameters η and ω by, for instance, a speckle-laser-interferometry technique and juxtaposing the data with relations (61) and (62) and the temperature distribution $\theta(r,t)$, one can retrieve some a posteriori information about the initial stress–strain state.

Obviously, the problem of retrieving the initial 2-D strain field directly from the integral equations (61) and (62) is ill-posed for several reasons. In particular, for local heating the measured parameters η and ω are not susceptible to the strains beyond the heating area. This can be overcome by scanning the heating spot on the object and measuring the parameters η and ω for each position of the heating spot. This allows to increase substantially the body of a posteriori information.

The inverse problem can be easily regularized when the heating-spot radius is small enough, as the initial strain components, ε_{ij}, can be considered as invariable in the heating area. Then, for the known temperature field $\theta(r,t)$, Eqs. (61) and (62) establish the finite relations connecting the initial strain components with the measured informative parameters. This means that for each heating spot we can write down two relations, connecting the main values ε_1 and ε_2 of the initial strain tensor in the area of this heating spot with the disturbed thermoelastic field parameters:

$$\eta(r,\varphi,t) = \frac{2\beta_2}{\lambda+2\mu}\left[\varepsilon_1 A_1(r,\varphi,t) + \varepsilon_2 A_2(r,\varphi,t)\right], \quad \omega(r,\varphi,t) = \frac{\beta_2}{2\mu}(\varepsilon_1-\varepsilon_2) A_3(r,\varphi,t), \tag{70}$$

where $A_i(r,\varphi,t)$ with $i=1,2,3$ are parameters that can be determined from the known temperature field of the disturbance:

$$A_1(r,\varphi,t) = \int_0^{2\pi}\int_0^{\infty}\left(\cos 2\varphi' \frac{\partial\theta}{\partial r'} + \sin^2\varphi' \frac{\partial}{\partial r'} r' \frac{\partial\theta}{\partial r'}\right) G\left(r,\varphi,r',\varphi'\right) \mathrm{d}r'\mathrm{d}\varphi'$$

$$A_2(r,\varphi,t) = \int_0^{2\pi}\int_0^{\infty}\left(\cos^2\varphi' \frac{\partial}{\partial r'} r' \frac{\partial\theta}{\partial r'} - \cos 2\varphi' \frac{\partial\theta}{\partial r'}\right) G\left(r,\varphi,r',\varphi'\right) \mathrm{d}r'\mathrm{d}\varphi'$$

$$A_3(r,\varphi,t) = \int_0^{2\pi}\int_0^{\infty} \sin 2\varphi \left(-\frac{\partial\theta}{\partial r'} + r'\frac{\partial^2\theta}{\partial r'^2}\right) G\left(r,\varphi,r',\varphi'\right) \mathrm{d}r'\mathrm{d}\varphi'$$

Thus, by measuring at some time moment t_1 the parameters η_1 and ω_1 at some point (r_1,φ_1) within the heating spot, one can solve the equations of (70) and find the main values ε_1 and ε_2 of the initial strain tensor in the area of this heating spot

$$\varepsilon_1 = \frac{\frac{\lambda+2\mu}{2\beta_2}\eta_1 A_3^1 + \frac{2\mu}{\beta_2}\omega_1 A_2^1}{A_3^1\left(A_1^1 + A_2^1\right)}, \quad \varepsilon_2 = \frac{\frac{2\mu}{\beta_2}\omega_1 A_1^1 - \frac{\lambda+2\mu}{2\beta_2}\eta_1 A_3^1}{A_3^1\left(A_1^1 + A_2^1\right)}. \tag{71}$$

Here $A_i^1 = A_i(r_1,\varphi_1,t_1), \quad i=1,2,3$.

As to the main directions of the initial strain tensor ε_{ij}, these can be found directly from the plots showing the dependencies of the parameters η and ω on the azimuth (see Fig. 1).

The parameters η and ω can be measured several times at $t_1, t_2, \ldots, t_m$ during the heating-and-cooling process at different points $(r_1,\varphi_1), (r_2,\varphi_2), \ldots, (r_m,\varphi_m)$ within the heating spot. Using the obtained data (η_1,ω_1), $(\eta_2,\omega_2), \ldots, (\eta_m,\omega_m)$ together with (70), one can obtain an overdetermined system of equations, which can be solved by a least-squares method. In such a way, the influence of random errors, unavoidable for such measurements, can be reduced.

Another application of the developed model can be defined using the relations (68) and (69) for the case of homogeneous heating. To measure the functions $\eta(x_1,x_2)$ and $\omega(x_1,x_2)$, we have the problem of determining the displacement components $u_1(x_1,x_2)$, $u_2(x_1,x_2)$ of the initial strain–stress state from the system of differential equations

$$\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = \frac{\lambda+2\mu}{(\beta_1+\beta_2)\theta}\left(\eta(x_1,x_2) - H_\eta\right), \quad \frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1} = \frac{\mu}{\beta_2\theta}\left(\omega(x_1,x_2) - H_\omega\right), \tag{72}$$

where the harmonic functions $H_\eta(x_1,x_2)$ and $H_\omega(x_1,x_2)$ should be chosen such as to satisfy all boundary and interface conditions.

10 Conclusion

The influence of the initial stress–strain state on thermoelastic processes induced in a prestressed solid by external heating, is accounted for in a developed mathematical model through thermoelastic constitutive equations, i.e., through dependencies of the elastic modulus and temperature-expansion coefficients on the initial strain components. Applying this approach for the case of small thermoelastic disturbances, we have obtained linear dynamic equations for the displacements. The coefficients of these equations are dependent on the initial strain components. The mathematical model describes temperature stresses and thermoelastic strains and displacements in piecewise homogeneous bodies in which stresses, caused by external loading, and residual stresses, caused by deformation incompatibility, occurring on the interface boundaries, act.

The solutions of pertinent boundary-value problems, formulated within the framework of the model, establish relationships between parameters of the thermoelastic process and the initial stress–strain state. Parameters of the processes, excited in the object by a known external thermal disturbance, can be measured. On the basis of these measurements, data and an appropriate boundary-value problem describing the applied-heating process, one can judge the initial stress–strain state of the object. Thus, the model can be used to develop nondestructive methods for determining the initial stress–strain state of prestressed objects.

To achieve this, an applicable technique for high-precision measuring of non-stationary thermoelastic displacements, and/or strains on the object surface during the process of its heating-and-cooling, should be developed. In this connection the method of electronic speckle-shearing interferometry looks very promising. Examples of the application of this precision measurement technique for stress-state parameter determination are given. For instance, the digital speckle-pattern interferometry method is applied to measure surface displacements when the hole-drilling method for the determination residual stresses is implemented [2]. This measurement technique can probably be accommodated to measure the surface thermoelastic strain parameters under local heating of the body.

Simple examples of local and homogeneous heating, considered in the paper, show how the developed model can be applied to retrieve parameters of the initial stress–strain state of the object from given parameters of thermoelastic strains at the surface, stimulated by the heating. The full-field retrieval of the stress–strain state of the object using thermoelastic processes and the digital speckle-pattern interferometry technique is possible through solving an appropriate inverse problem, stated within the framework of the developed mathematical model. The object geometry, structure and techniques of data collection should be accounted for in such an inverse problem.

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